Complete Characterization of Tractable Constraint Languages

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Overview

- Nonuniform CSPs
- Restrictive in practice
- Ubiquitous in TCS, approximation
- Complexity classification, dichotomy
- Algebraic approach: proves dichotomy, used in other areas
- Algorithms

The Problem

Setup

- Fixed finite domain

    (we will use some `derivative’ domains, though)

- Constraint relations are from a fixed set Γ, constraint language

- Relations are given explicitly (by a list of tuples)

- **CSP(Γ)**

What Would We Like to Know?

Complexity classification of nonuniform CSPs:

For each Γ, what is the complexity of **CSP(Γ)**?

Generalized Satisfiability

Regular **SAT**: Decide whether a given CNF is satisfiable

Generalized SAT: Decide whether a given conjunction of predicates on $\{0,1\}$ is satisfiable

$$(x \vee \bar{y} \vee z) \wedge (y \oplus \bar{t}) \wedge (t \neq u)$$

SAT is NP-complete

Gen SAT: depends on what predicates are allowed

Generalized Satisfiability II

Theorem (Schaefer, 1978**)**
Let Γ be a Boolean constraint language. Then **CSP(Γ)** is solvable in polynomial time if and only if one of the following conditions holds

(1) Γ is 0- or 1-valid
(2) Γ is Horn or anti-Horn
(3) Γ is binary (**2-SAT**)
(4) Γ is affine

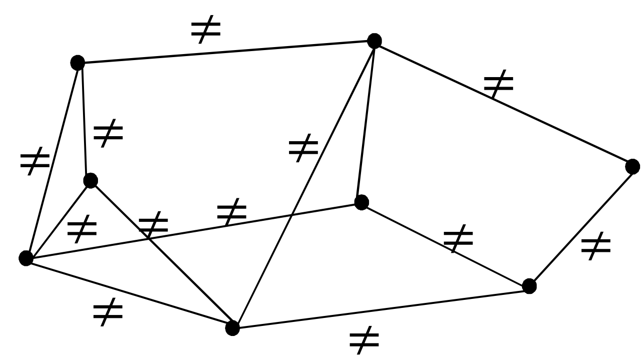
Otherwise **CSP(Γ)** is NP-complete

k- and H-Coloring

k-Coloring: **CSP($\neq$)**

Instance: A graph

$G=(V,E)$



Objective: Is it k-colorable?

H-Coloring: Replace ≠ with the edge relation of a graph H

Theorem (Hell,Nesetril, 1989**)**
The **H-Coloring** problem is polytime iff H has a loop or is bipartite. Otherwise it is NP-complete.

CSP and Logic

Fagin's Theorem: A problem belongs to NP iff it is expressible in Existential Second Order Logic

2-Coloring:

$$\exists R, B \left(\forall x, y \left(E(x,y) \rightarrow \Big(\big(R(x) \wedge B(y)\big) \vee \big(B(x) \wedge R(x)\big) \Big) \right) \wedge [R, B \text{ is partition}] \right)$$

CSP and Logic: MMSNP

MMSNP (Monotone Monadic Strict NP):

ESO formulas satisfying certain 3 syntactic conditions

Theorem (Feder,Vardi, 1993; Kun, 2013**)**
A problem is expressible in MMSNP iff it is polytime reducible to a CSP

If any of the 3 conditions is removed, can express the entire NP

Dichotomy Conjecture

Feder/Vardi, 1993

Dichotomy Conjecture:

For every Γ the problem **CSP(Γ)** is either solvable in poly time, or is NP-complete

CSP and Logic: Datalog

Datalog is `logic language' simulating the `least fixed point' operator

P(x,y) :- E(x,y)
P(x,y) :- P(x,z), E(z,t), E(t,y)
R(x)   :- P(x,x)

Datalog gives CSPs solvable by local propagation algorithms

Barto-Kozik, B.:   For non-uniform CSPs being solvable by Datalog is equivalent to a nice algebraic condition

Algebraic Approach

Invariants and Polymorphisms

Definition Relation R is invariant w.r.t. an n-ary operation f (or f is a polymorphism of R) if, for any $\bar{a}_1, \ldots, \bar{a}_n \in R$ the tuple obtained by applying f coordinate-wise belongs to R

Pol(Γ) denotes the set of all polymorphisms of relations from Γ

Theorem (Jeavons et al., 1998**)**
If Pol(Γ) $\subseteq$ Pol(Δ), then
CSP(Δ) is polytime reducible to **CSP(Γ)**

Polymorphisms: AntiHorn

Consider $R \in \Gamma_{antiHorn}$, say, $(x_1 \wedge x_2) \rightarrow x_3$

Then operation $min(x, y) = x \wedge y$ is a polymorphism of R

Indeed, take $(x_1, x_2, x_3), (y_1, y_2, y_3) \in R$, that is,
$(x_1 \wedge x_2) \rightarrow x_3$ and $(y_1 \wedge y_2) \rightarrow y_3$

Then we need to check that

$$\big((x_1 \wedge y_1) \wedge (x_2 \wedge y_2)\big) \rightarrow (x_3 \wedge y_3)$$

Polymorphisms: 2-SAT

Consider binary $R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Then operation $m(x, y, z)$:

$m(x, x, y) = m(x, y, x) = m(y, x, x) = x$ is a polymorphism of R (majority operation)

Indeed, apply m to the pairs of R, that is,

$$m \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Near-Unanimity function:

$$n(x, x, \ldots, x, y) = \cdots = n(y, x, \ldots, x) = x$$

Polymorphisms: Affine CSP

Consider $R \in \Gamma_{aff}$, it is the set of solutions of a system

$A \cdot \vec{x} = \vec{b}$

Then operation $f(x, y, z) = x - y + z$ is a polymorphism of R (affine operation)

Indeed, take $\vec{x}, \vec{y}, \vec{z} \in R$, that is, $A \cdot \vec{x} = A \cdot \vec{y} = A \cdot \vec{z} = \vec{b}$

Then

$$A \cdot (\vec{x} - \vec{y} + \vec{z}) = A \cdot \vec{x} - A \cdot \vec{y} + A \cdot \vec{z} = \vec{b} - \vec{b} + \vec{b} = \vec{b}$$

Polymorphisms: Schaefer Revisited

Theorem (Jeavons et al., 1997-9**)**
Let Γ be a constraint language over $\{0,1\}$. Then **CSP(Γ)** is polytime iff Γ has one of the following polymorphisms: constant function, $\vee$, $\wedge$, majority, or affine.
Otherwise it is NP-complete.

Polymorphisms: Tractability

Theorem (Jeavons et al., 1997-9; B., 2003-4**)**
If Γ has one of the following polymorphisms, then **CSP(Γ)** is polytime:
(1) Binary commutative operation $f(x, y) = f(y, x)$
(2) NU operation
(3) Maltsev operation $m(x, x, y) = m(y, x, x) = y$

Algebraic Dichotomy

- The algebraic approach can be further developed to make use of universal algebra

Algebraic Dichotomy Conjecture (B.,Jeavons,Krokhin., 2000**)**
If **CSP(Γ)** is polytime if and only if Γ has a `nontrivial' polymorphism. Otherwise it is NP-complete

`Nontrivial' polymorphisms can be characterized in many ways.

For instance, Γ has a such a polymorphism iff it has a weak NU:

$$n(x, x, \ldots, x, y) = \cdots = n(y, x, \ldots, x)$$

Dichotomies: Small Domains

The Dichotomy Conjecture holds if Γ is a constraint language on

- 2-element set (Schaefer, 1978)
- 3-element set (B., 2002)
- 4-element set (Marcovic, 2010)
- 5-element set (Zhuk, 2015)
- 7-element set (Zhuk, 2016)
- 9-element set (Zhuk, 2016)

Dichotomies: Conservative CSPs

Γ is said to be conservative if it contains every unary relation

Theorem (B., 2003**)**
The Algebraic Dichotomy Conjecture holds for conservative languages.

Two Algorithms: Local Propagation

Γ is said to have bounded width if is solved by local propagation (or by Datalog)

Theorem (Barto,Kozik, 2008**)**
Γ has bounded width iff it has two weak NU polymorphisms of different arity

A variety of local propagation techniques are equivalent in this case

Two Algorithms: Few Subpowers

Γ is said to have few subpowers if for any instance of **CSP(Γ)** there is a polynomial size representation of the solution space

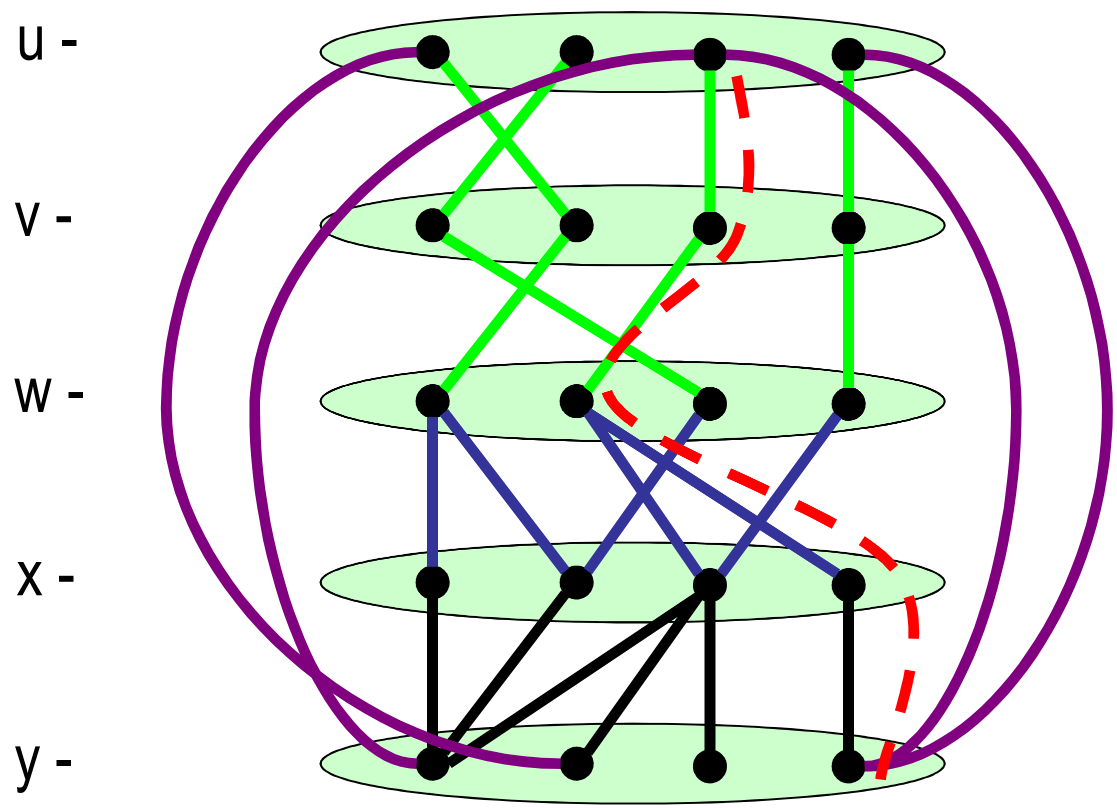
Theorem (Idziak et al., 2010**)**
Γ has few subpowers iff it has an edge polymorphism, and **CSP(Γ)** is polytime in this case

General Dichotomy

Theorem (B., 2017; Zhuk, 2017**)**
CSP(Γ) is polytime iff it has a weak NU polymorphism.
Otherwise it is NP-complete.

The Algorithm

Constraint Satisfaction Problem



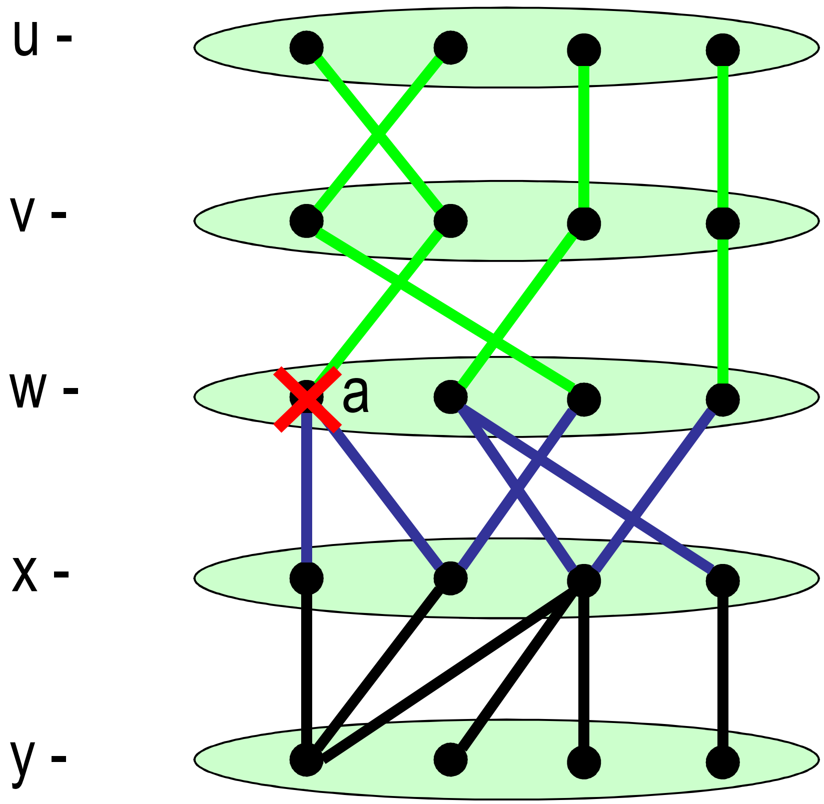
$Q(u,v,w)$

$R(w,x)$

$R(x,y)$

$S(y,u)$

Eliminating an Element



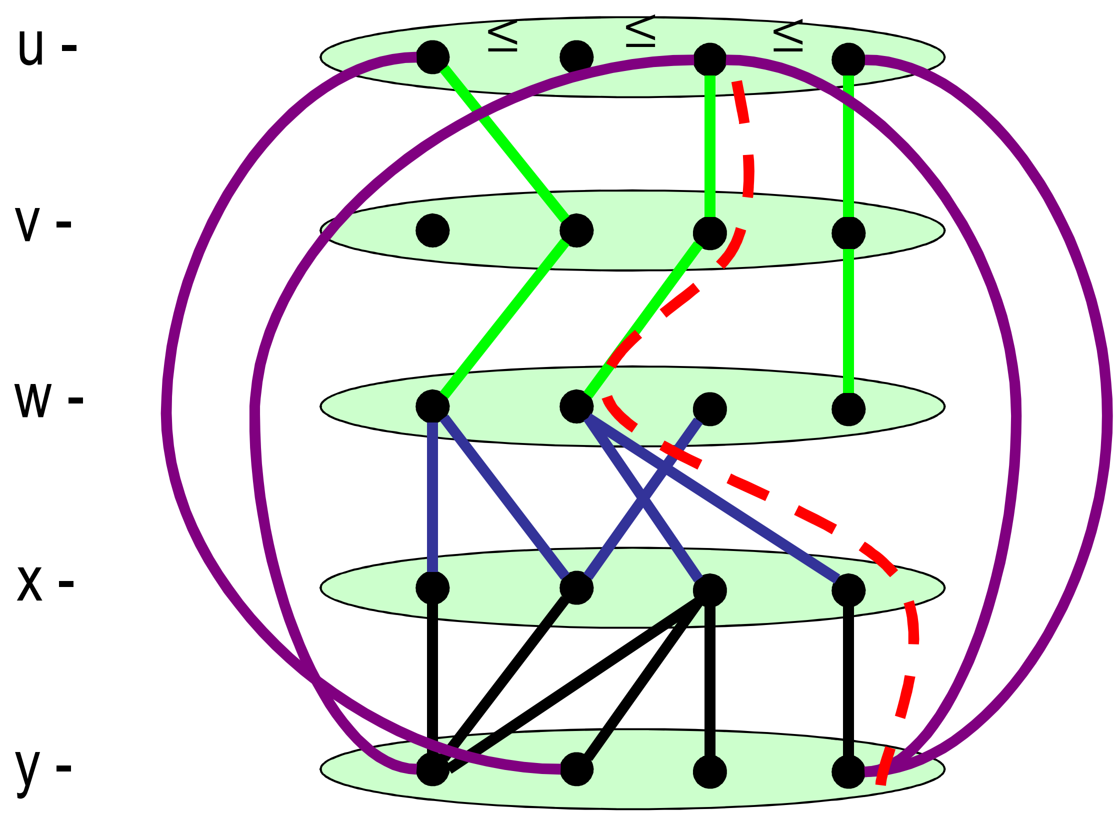
Is a a part of any solution?
No? Remove it!

If a IS a part of a solution, is there a solution that doesn't involve a?
Yes? Remove a!

Note that this procedure restricts the set D_v of possible values of a variable

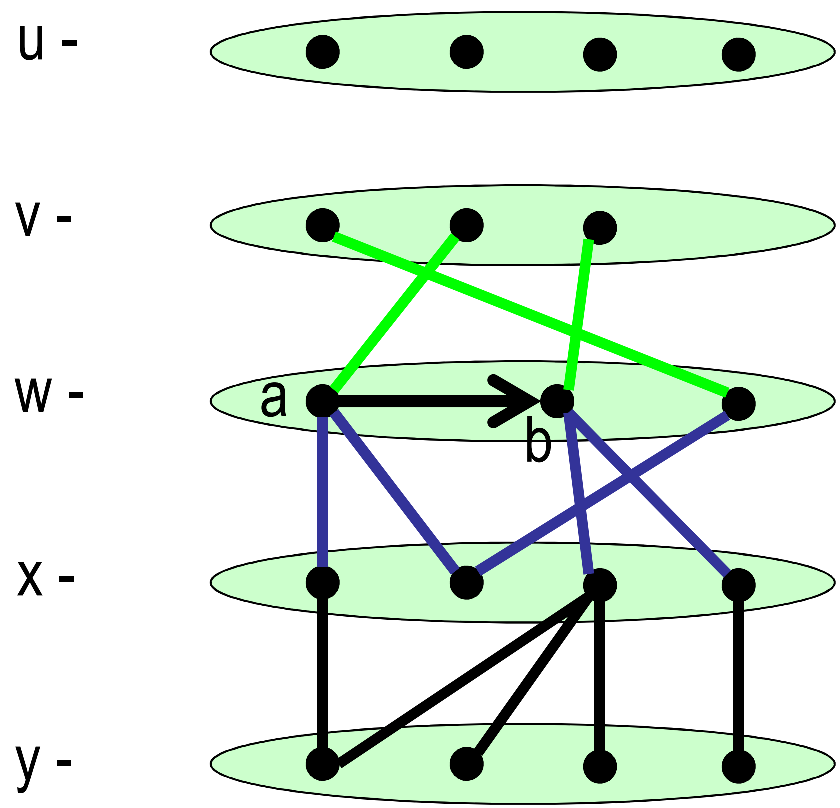
Local Propagation: Maximum

Suppose the domain is ordered, consider operation max



The tuple consisting maximal elements in each coordinate of a relation R belongs to R

Eliminating an Element II



For any solution using a there is a solution using b

Idea:
Establish sufficiently high level local consistency, then find such a redundant element

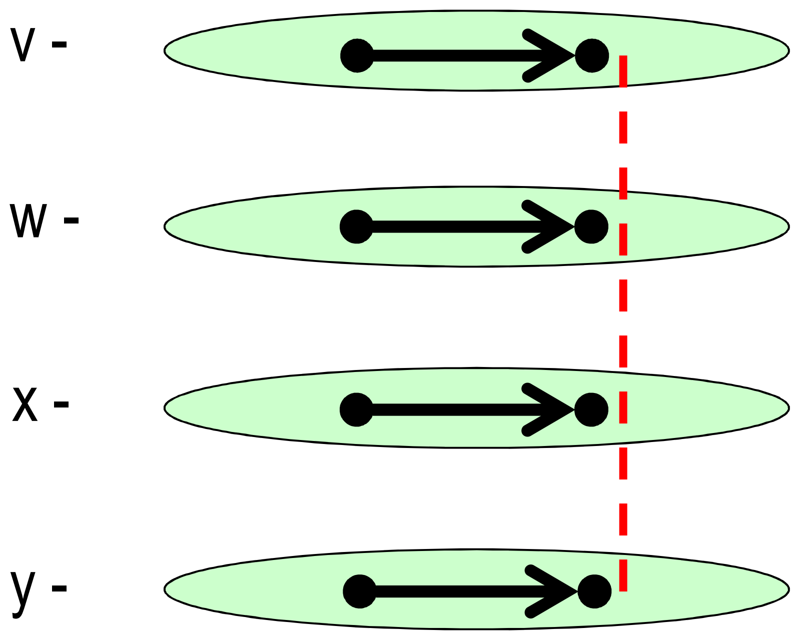
Doesn't Work!

The Method

- Identify `base case' problems solved by existing algorithms, reduce an arbitrary problem to the `base case'
- If the problem is not `base case'
    - Subdivide into polynomially many subproblems
    - Solve them recursively
    - Then either conclude that the problem has a solution,
    - or reduce every `bad' domain by at least 1 element

Semilattice Pairs

a, b is a semilattice pair if there is a polymorphism f such that $f(a,b) = f(b,a) = f(b,b) = b$ and $f(a,a) = a$



Local consistency + semilattice pairs is still not enough

Semilattice Free Languages

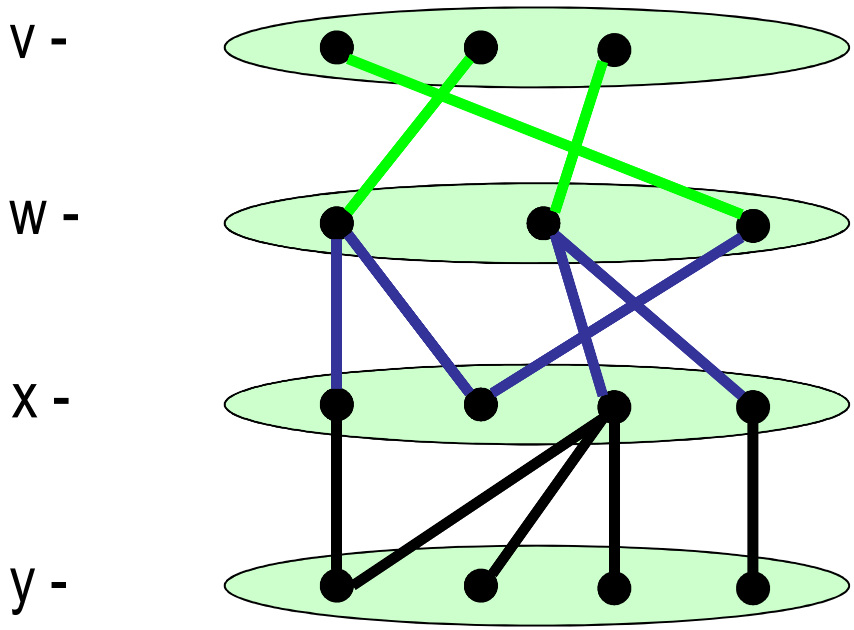
Language Γ is semilattice free if none of its domains has a semilattice pair

Theorem (Idziak et al., 2010 + B., 2016**)**
If Γ is semilattice free, then **CSP(Γ)** is polytime solvable

The Base Case

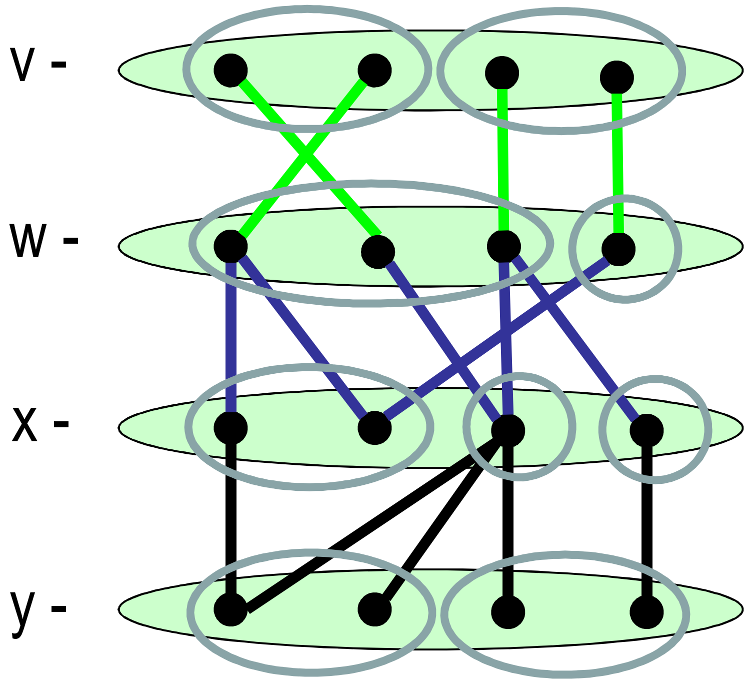
The `base case' is the problems whose domains are all semilattice free

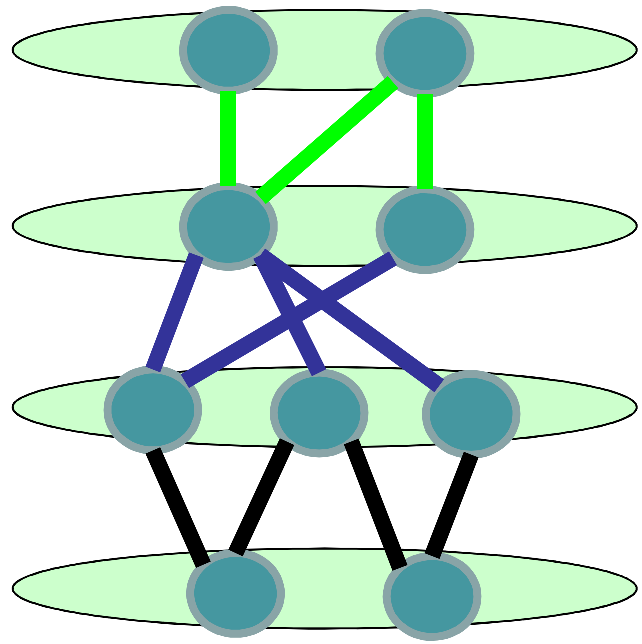
Minimal Instance



Instance is minimal if every tuple of every constraint can be extended to a solution

Factor Instance





I/μ

μ_v, μ_w, ... are partitions of the respective domains – have to be congruences of Γ

Block-Minimality

Instance $I = (V, \mathcal{C})$
Every domain comes in one of the two types:
- noncentral
- central

μ_v some congruence of the domain of v such that μ_v is equality for semilattice free domains

For variable v, congruences $\alpha \leq \beta$ of the domain of v,
$W = W(v, \alpha, \beta)$
I_W is the instance restricted to W

Block-Minimality II

Block-minimality: For every $W = W(v, \alpha, \beta)$ and $\langle s, R \rangle$

- if $v \in V$ is central then I_W/μ is minimal
- if $v \in V$ is non-central then I_W is minimal

Block-Minimality Works

Theorem

For any locally consistent instance I there are congruences μ such that if

I/μ is block-minimal

then I can be transformed to I' such that

- every domain containing a semilattice pair is reduced by at least 1 element;
- I' has a solution if and only if I does

Establishing Block-Minimality

If $W = W(v, Q, T)$ is central, then every domain of I_W/μ, is smaller than the original domains

Theorem

If $W = W(v, Q, T)$ is non-central then I_W can be decomposed into a constant number of instances over smaller domains

Open Problems

- Polymorphism oblivious algorithms and the Meta-problem

- Finer complexity classification

More Algebraic Approach

Algebraic approach is used for other constraint problems

- Polymorphism: decision, counting, enumeration, cardinality constraints, quantified, conjunctive queries, logic equivalence and minimization, social choice, etc.
- Valued CSPs: weighted clones
- Holant problems, partition functions: holant clones and functional clones
- Promise CSP: minions
- Many other problems: partial polymorphisms

Thank You!